

Free Motion of a Sphere in a Rotating Liquid parallel to the Axis of Rotation.

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§ 1. The motion is supposed to be a small disturbance from one of uniform rotation like a rigid body, small motion being defined to be such that the squares and products of velocity and vorticity components may be neglected in the expressions for acceleration. The system is supposed free from bodily forces, and is initially disturbed from relative rest by a motion suddenly communicated to the sphere.

The pressure intensity of the liquid consists of two parts, one depending only on the distance from the axis, and the other on the disturbed motion. If at any instant the sphere is moving parallel to the axis of rotation, and the disturbed motion of the liquid is symmetrical with respect to a line through the centre of the sphere parallel to the axis of rotation, the motional part of the pressure intensity will also be symmetrical with respect to this line, and its resultant effect on the sphere will be to produce an acceleration parallel to the axis of rotation. Now the positional part of the pressure intensity would maintain any portion of the liquid in relative rest, and will consequently maintain the sphere in relative rest, providing the density of the sphere is equal to that of the liquid, which we shall suppose to be the case.

The initial disturbed motion of the liquid will be irrotational, since the effects of rotation take time to develop, and if the initial relative motion of the sphere be parallel to the axis of rotation, the initial disturbed motion of the liquid will have the symmetrical character above specified.

It follows that the sphere will continue to move parallel to the axis of rotation, and the disturbed motion of the liquid will continue to be symmetrical with respect to the line of the sphere's relative motion, which in the sequel is called the axis of the sphere.

The method of discussion will be to assume the particular type of motion indicated above, and then to show that all the conditions can be satisfied by making it a particular function of the time. This leads to a perfectly definite mathematical problem, the solution of which gives the following results:—

The sphere oscillates about a point on its axis, the distance of which, from the initial position of the centre of the sphere, is proportional to the

velocity of projection; the amplitude of the oscillation tends rapidly to zero, but the period tends to a constant value which is half that of the rotation of the undisturbed liquid.

With regard to the velocity of the liquid a general expression has been obtained for its value at any point, reducing to simple expressions in particular cases. These have been examined.

On the axis of the sphere the velocity of the liquid is always less than the initial velocity of projection of the sphere and tends to zero at infinity. In the neighbourhood of the sphere the velocity oscillates about the zero value with a period which is ultimately constant and equal to that of the sphere's oscillation; the amplitude also tends to a value which is constant at a particular point, but there is an increasingly large gradient of velocity along the axis.

On the equatorial plane of the sphere it appears that the disturbance is practically confined to the sphere's immediate neighbourhood. Away from the sphere the velocity of the liquid rapidly tends to zero, oscillating about the zero value with decreasing amplitude and a period which is ultimately constant at a particular point. At infinity this period becomes that of the ultimate oscillation of the sphere, but increases without limit as the sphere is approached. The velocity gradient over the plane also increases without limit.

Finally the velocity of the liquid tangential to the sphere rapidly approaches a value which is constant in magnitude at any point, varying from zero at the poles to a maximum at the equator, the direction, however, changes with constant angular velocity which has a maximum value at the poles and is zero at the equator. Here again there is an unlimited increase in velocity gradient, in this case along a meridian of the sphere.

However, since the gradients of the velocity of the liquid over the equatorial plane and of the transverse velocity along a meridian of the sphere ultimately increase without limit, parts of this solution are not applicable, for a stage is reached after which the components of vorticity cannot be considered as small. Consequently the physical assumptions underlying the mathematical treatment, viz., those of small motion, are violated; in other words, the solution will only represent the true state of the liquid for a restricted time.

The question of the ultimate physical state thus remains unanswered. J. Proudman* showed that a small steady disturbance was impossible, but then G. I. Taylor† obtained a solution of the general equations of steady

* "On the Motion of Solids in a Liquid Possessing Vorticity," 'Roy. Soc. Proc.,' A, vol. 92, 1916.

† 'Proc. Camb. Phil. Soc.,' vol. 20, Part III.

symmetrical motion about a sphere, which does not satisfy the equations of small motion. Whether or not Taylor's solution represents the ultimate state in the problem of the present paper is a question for further investigation.

The work was suggested to me by Prof. J. Proudman, to whom I am greatly indebted for his interest and advice; it is an extension of some of his own work (*loc. cit.*). It is proposed in a future paper to give the corresponding results when the sphere is projected at right angles to the axis of rotation, part of the initial work for which has already been effected. A combination of the two cases will give the small general motion of a sphere projected with given initial velocity.

Mathematical Problem.

§ 2. We shall use Cartesian axes *Oxyz* rotating about *Oz* with constant angular velocity ω , called the primary axes. Let u, v, w denote the components of the velocity at x, y, z relative to these axes. The equation of continuity for a uniform liquid gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1)$$

and the boundary condition may be written

$$lu + mv + nw = nV, \quad (2)$$

l, m, n being the direction cosines of the normal to the sphere and V the velocity of the sphere. These relations being kinematical are independent of the fact that the axes are rotating.

The dynamical equations, in the case of small motion under no forces other than the pressure of the liquid, reduce to

$$\left. \begin{aligned} \frac{\partial u}{\partial t} - 2\omega v &= -\frac{\partial P}{\partial x} \\ \frac{\partial v}{\partial t} + 2\omega u &= -\frac{\partial P}{\partial y} \\ \frac{\partial w}{\partial t} &= -\frac{\partial P}{\partial z} \end{aligned} \right\}, \quad (3)$$

where $P = p/\rho - \frac{1}{2}\omega^2(x^2 + y^2), \quad (4)$

the letters p and ρ having their usual signification.

Take a set of axes, having their origin always at the centre of the sphere (the secondary axes), and let these secondary axes be always parallel to the primary axes.

In equations (1)–(4) the actual values of x, y, z are only required in the

term $\frac{1}{2}\omega^2(x^2+y^2)$ in (4), since in $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ we have t constant. Hence if x, y, z now refer to the secondary axes, the equations (1), (2), (3) remain, but

$$P = p/\rho - \frac{1}{2}\omega^2(x_0^2 + y_0^2),$$

x_0, y_0, z_0 , referring to the primary axes.

The term $\frac{1}{2}\rho\omega^2(x_0^2 + y_0^2)$ in

$$p = \rho P + \frac{1}{2}\rho\omega^2(x_0^2 + y_0^2)$$

would maintain a liquid sphere in relative rest, *i.e.*, would produce an acceleration $\omega^2(x_0^2 + y_0^2)^{\frac{1}{2}}$ towards the axis of rotation. It will therefore produce an equal acceleration in the solid sphere, and we may take

$$p = \rho P$$

as the effective pressure intensity in producing the acceleration of the sphere other than centripetal.

Now assume that the functions u, v, w, V , and P may be expressed as power series in time. This proves to be legitimate, since the results obtained show these series to be highly convergent for finite values of t . Let

$$u = \sum_{s=0}^{\infty} u_s \frac{(2\omega t)^s}{s!}, \quad v = \sum_{s=0}^{\infty} v_s \frac{(2\omega t)^s}{s!}, \quad w = \sum_{s=0}^{\infty} w_s \frac{(2\omega t)^s}{s!},$$

$$V = \sum_{s=0}^{\infty} c_s \frac{(2\omega t)^s}{s!}, \quad P = 2\omega \sum_{s=0}^{\infty} P_s \frac{(2\omega t)^s}{s!},$$

where u_s, v_s, w_s, c_s , and P_s are functions of x, y, z only, the initial state being given by u_0, v_0, w_0, c_0 .

Substituting these expressions in the equations (1), (2), (3), we obtain

$$\frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial y} + \frac{\partial w_s}{\partial z} = 0, \quad s = 0, 1, 2, \dots, \quad (5)$$

since the equation (1) is true for all time; the boundary condition gives

$$lu_s + mv_s + nw_s = mc_s, \quad s = 0, 1, 2, \dots,$$

and from the dynamical equations

$$\left. \begin{aligned} u_{s+1} - v_s &= -\partial P_s / \partial x \\ v_{s+1} + u_s &= -\partial P_s / \partial y \\ w_{s+1} &= -\partial P_s / \partial z \end{aligned} \right\}, \quad s = 0, 1, 2, \dots, \quad (6)$$

from which

$$\left. \begin{aligned} u_{s+1} + u_{s-1} &= -\frac{\partial P_s}{\partial x} - \frac{\partial P_{s-1}}{\partial y} \\ v_{s+1} + v_{s-1} &= -\frac{\partial P_s}{\partial y} + \frac{\partial P_{s-1}}{\partial x} \\ w_{s+1} + w_{s-1} &= -\frac{\partial P_s}{\partial z} - \frac{\partial P_{s-2}}{\partial z} \end{aligned} \right\}, \quad s = 2, 3, 4, \dots, \quad (7)$$

which remain valid for $s = 1$ if we replace $\partial P_{-1}/\partial z$ by $-w_0$.

The velocity of the liquid at any point may now be expressed in terms of the sequence P_s , thus*

$$\begin{aligned} u - u_0\gamma_0(2\omega t) - v_0\gamma_1(2\omega t) \\ &= -\partial/\partial x \sum_{s=0}^{\infty} P_s\gamma_{s+1}(2\omega t) - \partial/\partial y \sum_{s=0}^{\infty} P_s\gamma_{s+2}(2\omega t), \\ v - v_0\gamma_0(2\omega t) + u_0\gamma_1(2\omega t) \\ &= -\partial/\partial y \sum_{s=0}^{\infty} P_s\gamma_{s+1}(2\omega t) + \partial/\partial x \sum_{s=0}^{\infty} P_s\gamma_{s+2}(2\omega t), \\ w - w_0 &= -\partial/\partial z \sum_{s=0}^{\infty} P_s(2\omega t)^{s+1}/(s+1)! \end{aligned}$$

where
$$\gamma_s(z) = \sum_{n=0}^{\infty} (-1)^n z^{s+2n}/(s+2n)!$$

and we now concentrate our attention on the sequence P_s .

Inside the liquid we have, from (5) and (7),

$$\nabla^2 P_s = -\partial^2 P_{s-2}/\partial z^2, \quad s = 1, 2, 3 \dots,$$

and from (6)

$$\nabla^2 P_0 = 0,$$

while at the boundary

$$\frac{\partial P_s}{\partial n} = -nc_{s+1} - nc_{s-1} - n \frac{\partial P_{s-2}}{\partial z}, \quad s = 1, 2, 3, \dots,$$

on account of the symmetrical character already referred to, with

$$\partial P_0/\partial n = -nc_1.$$

The initial motion being irrotational, we may let P_{-1} denote the velocity potential, so that

$$u_0 = -\partial P_{-1}/\partial x, \quad v_0 = -\partial P_{-1}/\partial y, \quad w_0 = -\partial P_{-1}/\partial z,$$

and

$$\nabla^2 P_{-1} = 0, \quad \partial P_{-1}/\partial n = -nc_0.$$

The value of c_0 , the initial velocity of the sphere, determines the magnitude of the subsequent motion; in the sequel this quantity, c_0 , is denoted by W .

The resultant pressure on the sphere is in the direction Oz and equal to

$$-\rho \iint (P)_{r=a} \cos \theta \, dS$$

taken over the sphere, where r, θ, ϕ are the spherical polar co-ordinates of the element dS referred to the centre of the sphere as origin, $\theta = 0$ denoting the z axis and $\phi = 0$ the plane xOz , a being the radius of the sphere. This becomes

$$-2\omega\rho \sum_{s=0}^{\infty} \frac{(2\omega t)^s}{s!} \iint (P_s)_{r=a} \cos \theta \, dS,$$

* Proudman, *loc. cit.*

and the equations of motion of the sphere reduce to

$$\sum_{s=0}^{\infty} \frac{(2\omega t)^s}{s!} \iint (P_s)_{r=a} \cos \theta \, dS = -\frac{4}{3}\pi a^3 \sum_{s=1}^{\infty} c_s \frac{(2\omega t)^{s-1}}{(s-1)!}$$

so that

$$c_{s+1} = -\iint (P_s)_{r=a} \cos \theta \, dS / (\frac{4}{3}\pi a^3), \quad s = 0, 1, 2, \dots, \quad (8)$$

giving the functions c_s in terms of the sequence P_s .

Assuming the liquid to be at relative rest at infinity, the boundary conditions may now be written.

$$\left(\frac{\partial P_s}{\partial r}\right)_{r=a} = \frac{\cos \theta}{\frac{4}{3}\pi a^3} \iint (P_s + P_{s-2})_{r=a} \cos \theta \, dS - \cos \theta \left(\frac{\partial P_{s-2}}{\partial z}\right)_{r=a}, \quad s = 2, 3, \dots,$$

and

$$\begin{aligned} \left(\frac{\partial P_1}{\partial r}\right)_{r=a} &= \frac{\cos \theta}{\frac{4}{3}\pi a^3} \iint (P_1)_{r=a} \cos \theta \, dS - W \cos \theta - \cos \theta \left(\frac{\partial P_{-1}}{\partial z}\right)_{r=a}, \\ \left(\frac{\partial P_0}{\partial r}\right)_{r=a} &= \frac{\cos \theta}{\frac{4}{3}\pi a^3} \iint (P_0)_{r=a} \cos \theta \, dS. \end{aligned}$$

$$\text{Now} \quad \nabla^2 P_{-1} = 0, \quad (\partial P_{-1} / \partial r)_{r=a} = -W \cos \theta,$$

$$\text{hence} \quad P_{-1} = \frac{1}{2} W a (a/r)^2 \cos \theta$$

$$\text{and} \quad \nabla^2 P_0 = 0, \quad \left(\frac{\partial P_0}{\partial r}\right)_{r=a} = \frac{\cos \theta}{\frac{4}{3}\pi a^3} \iint (P_0)_{r=a} \cos \theta \, dS.$$

The function P_0 has to satisfy these two relations and vanish at infinity; this is impossible unless P_0 is zero. Generally

$$\nabla^2 P_{2s} = 0, \quad \left(\frac{\partial P_{2s}}{\partial r}\right)_{r=a} = \frac{\cos \theta}{\frac{4}{3}\pi a^3} \iint (P_{2s})_{r=a} \cos \theta \, dS,$$

$$\therefore \quad P_{2s} = 0.$$

Hence we require a sequence of functions P_{2s-1} ($s = 1, 2, \dots$), which vanish at infinity and satisfy

$$\nabla^2 P_{2s-1} = -\partial^2 P_{2s-3} / \partial z^2, \quad s = 2, 3, \dots, \quad (9)$$

inside the liquid and

$$\left(\frac{\partial P_{2s-1}}{\partial r}\right)_{r=a} = \frac{\cos \theta}{\frac{4}{3}\pi a^3} \iint (P_{2s-1} + P_{2s-3})_{r=a} \cos \theta \, dS - \cos \theta \left(\frac{\partial P_{2s-3}}{\partial z}\right)_{r=a}, \quad s = 2, 3, \dots, \quad (10)$$

together with

$$\nabla^2 P_1 = -\partial^2 / \partial z^2 \left\{ \frac{1}{2} W a (a/r)^2 \cos \theta \right\}, \quad (11)$$

and

$$\begin{aligned} \left(\frac{\partial P_1}{\partial r}\right)_{r=a} &= \frac{\cos \theta}{\frac{4}{3}\pi a^3} \iint (P_1)_{r=a} \cos \theta \, dS - W \cos \theta \\ &\quad - \cos \theta \left[\frac{\partial}{\partial z} \left\{ \frac{1}{2} W a \left(\frac{a}{r}\right)^2 \cos \theta \right\} \right]_{r=a}. \end{aligned} \quad (12)$$

The Solution.

§ 3. After an investigation which would take too much space to reproduce, the required sequence was found to be as follows:—

$$P_{2s-1} = Wa \{ (a/r)^2 Q_{2, 2s}(\cos \theta) + (a/r)^4 Q_{4, 2s}(\cos \theta) + \dots \\ \dots + (a/r)^{2m} Q_{2m, 2s}(\cos \theta) + \dots + (a/r)^{2s+2} Q_{2s+2, 2s}(\cos \theta) \}, \quad (13)$$

where, putting $\cos \theta = \mu$,

$$Q_{2m, 2s}(\mu) = (-1)^{s-m} \frac{3}{2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2s-1)}{2^s (m-1)! (s-m+1)!} \mu (1-\mu^2)^{s-m} \times \\ \left[\frac{2}{2m+1} \left\{ 1 - \frac{2s+1}{3} {}^{m-1}C_1 \mu^2 + \frac{2s+1}{3} \cdot \frac{2s+3}{5} {}^{m-1}C_2 \mu^4 - \dots \right. \right. \\ \left. \left. \dots + (-1)^{m-1} \frac{2s+1}{3} \cdot \frac{2s+3}{5} \dots \frac{2s+2m-3}{2m-1} \mu^{2m-2} \right\} \right. \\ \left. - \left\{ 1 - \frac{2s+1}{3} {}^mC_1 \mu^2 + \frac{2s+1}{3} \cdot \frac{2s+3}{5} {}^mC_2 \mu^4 - \dots \right. \right. \\ \left. \left. \dots + (-1)^m \frac{2s+1}{3} \cdot \frac{2s+3}{5} \dots \frac{2s+2m-1}{2m+1} \mu^{2m} \right\} \right], \quad (14)$$

for $s > 0$ and $m = 1, 2, \dots (s+1)$. A more compact form is given by

$$Q_{2m, 2s}(\mu) = (-1)^{s-m} \frac{3}{4} \cdot \frac{(1-\mu^2)^{s-m}}{(m-1)! (s-m+1)!} \times \\ \left(\frac{d}{d\mu^2} \right)^{s-1} \left\{ \frac{2\mu^{2s-1}}{2m+1} (1-\mu^2)^{m-1} - \mu^{2s-1} (1-\mu^2)^m \right\}, \quad (15)$$

and it will be noticed that

$$Q_{2s+2, 2s}(\mu) = (-1)^s \frac{3}{2} P_{2s+1}(\mu) / (2s+3),$$

where, in this case, $P_{2s+1}(\mu)$ denotes Legendre's function.

We shall now simply verify that these formulæ satisfy all the conditions.

Verification.

§ 4. Considering first the relation (9), the expression (13) satisfies this provided

$$\nabla^2 \{ (a/r)^{2m} Q_{2m, 2s} \} = -\partial^2 / \partial z^2 \{ (a/r)^{2m} Q_{2m, 2s-2} \}, \quad s = 2, 3, \dots, \\ i.e. (1-\mu^2) \frac{\partial^2 Q_{2m, 2s}}{\partial \mu^2} - 2\mu \frac{\partial Q_{2m, 2s}}{\partial \mu} + 2m(2m-1) Q_{2m, 2s} \\ = - \left[(1-\mu^2)^2 \frac{\partial^2 Q_{2m, 2s-2}}{\partial \mu^2} - (4m+3) \mu (1-\mu^2) \frac{\partial Q_{2m, 2s-2}}{\partial \mu} \right. \\ \left. + 2m \{ 2(m+1) \mu^2 - 1 \} Q_{2m, 2s-2} \right].$$

This relation is satisfied when $m = s+1$ since $Q_{2s+2, 2s}$ is Legendre's function of order $(2s+1)$, apart from a factor, and $Q_{2s+2, 2s-2}$ is zero. Otherwise let

$$Q_{2m, 2s} = (-1)^{s-m} \frac{3}{2} \frac{1 \cdot 3 \cdot 5 \dots (2s-1)}{2^s (m-1)! (s-m+1)!} Q_{2m, 2s},$$

then we have to show that

$$\begin{aligned} (2s-1) & \left[(1-\mu^2) \frac{\partial^2 q_{2m, 2s}}{\partial \mu^2} - 2\mu \frac{\partial q_{2m, 2s}}{\partial \mu} + 2m(2m-1) q_{2m, 2s} \right] \\ & = 2(s-m+1) \left[(1-\mu)^2 \frac{\partial^2 q_{2m, 2s-2}}{\partial \mu^2} - (4m+3)\mu(1-\mu^2) \frac{\partial q_{2m, 2s-2}}{\partial \mu} \right. \\ & \quad \left. + 2m\{2(m+1)\mu^2-1\} q_{2m, 2s-2} \right]. \end{aligned}$$

Now

$$\begin{aligned} q_{2m, 2s} & = (1-\mu^2)^{s-m} \left[\frac{2}{2m+1} \left\{ \mu - \frac{2s+1}{3} {}^{m-1}C_1 \mu^3 + \frac{2s+1}{3} \cdot \frac{2s+3}{5} {}^{m-1}C_2 \mu^5 - \dots \right. \right. \\ & \quad \left. \dots + (-1)^k \frac{2s+1}{3} \cdot \frac{2s+3}{5} \dots \frac{2s+2k-1}{2k+1} {}^{m-1}C_k \mu^{2k+1} + \dots \right. \\ & \quad \left. \dots + (-1)^{m-1} \frac{2s+1}{3} \cdot \frac{2s+3}{5} \dots \frac{2s+2m-3}{2m-1} \mu^{2m-1} \right\} \\ & \quad \left. - \left\{ \mu - \frac{2s+1}{3} {}^mC_1 \mu^3 + \dots + (1)^m \frac{2s+1}{3} \dots \frac{2s+2m-1}{2m+1} \mu^{2m+1} \right\} \right] \\ & = (1-\mu^2)^{s-m} \alpha_{2s} \text{ say.} \end{aligned}$$

$$\begin{aligned} \frac{\partial q_{2m, 2s}}{\partial \mu} & = -2(s-m)(1-\mu^2)^{s-m-1} \mu \alpha_{2s} \\ & \quad + (1-\mu^2)^{s-m} \left[\frac{2}{2m+1} \left\{ 1 - \frac{2s+1}{3} {}^{m-1}C_1 3\mu^2 \right. \right. \\ & \quad \left. \left. + \frac{2s+1}{3} \cdot \frac{2s+3}{5} {}^{m-1}C_2 5\mu^4 - \dots \right. \right. \\ & \quad \left. \left. \dots + (-1)^k \frac{2s+1}{3} \dots \frac{2s+2k-1}{2k+1} {}^{m-1}C_k (2k+1) \mu^{2k} + \dots \right\} \right. \\ & \quad \left. - \left\{ 1 - \frac{2s+1}{3} {}^mC_1 3\mu^2 + \dots \right\} \right] \\ & = -2(s-m)(1-\mu^2)^{s-m-1} \mu \alpha_{2s} + (1-\mu^2)^{s-m} \beta_{2s} \text{ say.} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 q_{2m, 2s}}{\partial \mu^2} & = 4(s-m)(s-m-1)(1-\mu^2)^{s-m-2} \mu^2 \alpha_{2s} \\ & \quad - 2(s-m)(1-\mu^2)^{s-m-1} \alpha_{2s} - 4(s-m)(1-\mu^2)^{s-m-1} \mu \beta_{2s} \\ & \quad + (1-\mu^2)^{s-m} \left[\frac{2}{2m+1} \left\{ -\frac{2s+1}{3} {}^{m-1}C_1 3 \cdot 2\mu \right. \right. \\ & \quad \left. \left. + \frac{2s+1}{3} \cdot \frac{2s+3}{5} {}^{m-1}C_2 5 \cdot 4\mu^3 - \dots \right. \right. \\ & \quad \left. \left. \dots + (-1)^k \frac{2s+1}{3} \dots \frac{2s+2k-1}{2k+1} {}^{m-1}C_k (2k+1) 2k \mu^{2k-1} + \dots \right\} \right. \\ & \quad \left. - \left\{ -\frac{2s+1}{3} {}^mC_1 3 \cdot 2\mu + \dots \right\} \right] \\ & = 2(s-m) \{ (2s-2m-1) \mu^2 - 1 \} (1-\mu^2)^{s-m-2} \alpha_{2s} \\ & \quad - 4(s-m)(1-\mu^2)^{s-m-1} \mu \beta_{2s} + (1-\mu^2)^{s-m} \gamma_{2s} \text{ say.} \end{aligned}$$

Hence

$$\begin{aligned}
 (1-\mu^2) \frac{\partial^2 q_{2m, 2s}}{\partial \mu^2} - 2\mu \frac{\partial q_{2m, 2s}}{\partial \mu} + 2m(2m-1) q_{2m, 2s} \\
 = (1-\mu^2)^{s-m-1} [\{ 2s(2s-4m+1)\mu^2 + 4m^2 - 2s \} \alpha_{2s} \\
 - 2(2s-2m+1)\mu(1-\mu^2)\beta_{2s} + (1-\mu^2)^2 \gamma_{2s}]
 \end{aligned}$$

and

$$\begin{aligned}
 (1-\mu^2)^2 \frac{\partial^2 q_{2m, 2s-2}}{\partial \mu^2} - (4m+3)\mu(1-\mu^2) \frac{\partial q_{2m, 2s-2}}{\partial \mu} \\
 + 2m \{ 2(m+1)\mu^2 - 1 \} q_{2m, 2s-2} \\
 = (1-\mu^2)^{s-m-1} [\{ 4s(s-1)\mu^2 - 2(s-1) \} \alpha_{2s-2} \\
 - (4s-1)\mu(1-\mu^2)\beta_{2s-2} + (1-\mu^2)^2 \gamma_{2s-2}];
 \end{aligned}$$

so that we are required to show that

$$\begin{aligned}
 (2s-1) [\{ 2s(2s-4m+1)\mu^2 + 4m^2 - 2s \} \alpha_{2s} \\
 - 2(2s-2m+1)\mu(1-\mu^2)\beta_{2s} + (1-\mu^2)^2 \gamma_{2s}] \quad (16)
 \end{aligned}$$

is identical with

$$\begin{aligned}
 2(s-m+1) [\{ 4s(s-1)\mu^2 - 2(s-1) \} \alpha_{2s-2} \\
 - (4s-1)\mu(1-\mu^2)\beta_{2s-2} + (1-\mu^2)^2 \gamma_{2s-2}]. \quad (17)
 \end{aligned}$$

The coefficient of μ^{2k+1} in (16) is ($k = 2, 3, \dots, m-2$)

$$\begin{aligned}
 (-1)^{k+1} \left[\frac{2s-1}{1} \cdot \frac{2s+1}{3} \cdot \dots \cdot \frac{2s+2k+1}{2k+3} \times \right. \\
 \left(\frac{2}{2m+1} {}^{m-1}C_{k+1} - {}^mC_{k+1} \right) (2k+3)(2k+2) \\
 + \frac{2s-1}{1} \cdot \dots \cdot \frac{2s+2k-1}{2k+1} \left(\frac{2}{2m+1} {}^{m-1}C_k - {}^mC_k \right) \times \\
 \{ -4m^2 + 2s + 2(2s-2m+1)(2k+1) + 2(2k+1)2k \} \\
 + \frac{2s-1}{1} \cdot \dots \cdot \frac{2s+2k-3}{2k-1} \left(\frac{2}{2m+1} {}^{m-1}C_{k-1} - {}^mC_{k-1} \right) \times \\
 \left. \{ 2s(2s-4m+1) + 2(2s-2m+1)(2k-1) + (2k-1)(2k-2) \} \right] \\
 = (-1)^{k+1} \frac{2s-1}{1} \cdot \frac{2s+1}{3} \cdot \dots \cdot \frac{2s+2k-1}{2k+1} \times \\
 \left\{ 4(s-m) {}^mC_k - {}^mC_{k-1} \frac{2(s-m+1)(2m+3)(m+1) - 4(m-k+1)}{k} \right\} \\
 = (-1)^{k+1} \frac{2s-1}{1} \cdot \frac{2s+1}{3} \cdot \dots \cdot \frac{2s+2k-1}{2k+1} 2(s-m+1) \times \\
 \{ 2 {}^mC_k - (2m+3) {}^{m+1}C_k \},
 \end{aligned}$$

and the coefficient of μ^{2k+1} in (17) is

$$\begin{aligned}
 & (-1)^{k+1} \left[2(s-m+1) \frac{2s-1}{3} \cdot \frac{2s+1}{5} \dots \frac{2s+2k-1}{2k+3} \times \right. \\
 & \quad \left(\frac{2}{2m+1} {}^{m-1}C_{k+1} - {}^mC_{k+1} \right) (2k+3)(2k+2) \\
 & \quad + 2(s-m+1) \frac{2s-1}{3} \dots \frac{2s+2k-3}{2k+1} \left(\frac{2}{2m+1} {}^{m-1}C_k - {}^mC_k \right) \times \\
 & \quad \{ 2(s-1) + (4s-1)(2k+1) + 2(2k+1)2k \} \\
 & \quad + 2(s-m+1) \frac{2s-1}{3} \dots \frac{2s+2k-5}{2k-1} \left(\frac{2}{2m+1} {}^{m-1}C_{k-1} - {}^mC_{k-1} \right) \times \\
 & \quad \left. \{ 4s(s-1) + (4s-1)(2k-1) + (2k-1)(2k-2) \} \right] \\
 & = (-1)^{k+1} 2(s-m+1) \frac{2s-1}{1} \cdot \frac{2s+1}{3} \dots \frac{2s+2k-1}{2k+1} \times \\
 & \quad \{ 2 {}^mC_k - (2m+3) {}^{m+1}C_k \},
 \end{aligned}$$

and it is easily verified that the coefficients of μ , μ^3 , μ^{2m+1} , μ^{2m+3} in (16) and (17) are also identical.

Examining now the case where $s = 1$, we have

$$\begin{aligned}
 \nabla^2 P_1 &= \nabla^2 W a \left\{ \frac{1}{4} (a/r)^2 \mu (3\mu^2 - 1) - \frac{3}{20} (a/r)^4 \mu (5\mu^2 - 3) \right\} \\
 &= \frac{3}{2} W/a (a/r)^4 \mu (3 - 5\mu^2) \\
 &= -\partial^2 / \partial z^2 \left\{ \frac{1}{2} W a (a/r)^2 \mu \right\}.
 \end{aligned}$$

It now remains to show that the assumed form (13) of P_{2s-1} satisfies the boundary conditions.

On $r = a$ we have

$$P_{2s-1} = W a \{ Q_{2, 2s}(\mu) + Q_{4, 2s}(\mu) + \dots + Q_{2m, 2s}(\mu) + \dots + Q_{2s+2, 2s}(\mu) \},$$

and, examining the series within the brackets, we have, putting $\mu^2 = \nu$

$$\begin{aligned}
 & Q_{2, 2s} + Q_{4, 2s} + \dots + Q_{2m, 2s} + \dots + Q_{2s+2, 2s} \\
 &= (-1)^{s-1} \frac{3}{4} \left[\frac{(1-\nu)^{s-1}}{s!} \left(\frac{d}{d\nu} \right)^{s-1} \left\{ \frac{2}{3} \nu^{(2s-1)/2} - \nu^{(2s-1)/2} (1-\nu) \right\} \right. \\
 & \quad - \frac{(1-\nu)^{s-2}}{1!(s-1)!} \left(\frac{d}{d\nu} \right)^{s-1} \left\{ \frac{2}{5} \nu^{(2s-1)/2} (1-\nu) - \nu^{(2s-1)/2} (1-\nu)^2 \right\} \\
 & \quad + \frac{(1-\nu)^{s-3}}{2!(s-2)!} \left(\frac{d}{d\nu} \right)^{s-1} \left\{ \frac{2}{7} \nu^{(2s-1)/2} (1-\nu)^2 - \nu^{(2s-1)/2} (1-\nu)^3 \right\} \\
 & \quad \dots - (-1)^m \frac{(1-\nu)^{s-m}}{(m-1)!(s-m+1)!} \times \\
 & \quad \left(\frac{d}{d\nu} \right)^{s-1} \left\{ \frac{2}{2m+1} \nu^{(2s-1)/2} (1-\nu)^{m-1} - \nu^{(2s-1)/2} (1-\nu)^m \right\} \\
 & \quad \left. \dots + (-1)^s \frac{(1-\nu)^{-1}}{s!} \left(\frac{d}{d\nu} \right)^{s-1} \left\{ \frac{2}{2s+3} \nu^{(2s-1)/2} (1-\nu)^s - \nu^{(2s-1)/2} (1-\nu)^{s+1} \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= (-1)^{s-1} 3 \left[\frac{\nu^{1/2} (1-\nu)^{s-1}}{(2s+1)(2s+3)} + \frac{\nu^{3/2} (1-\nu)^{s-2}}{(2s+1)(2s+3)} s^{-1} C_1 + \dots \right] \\
 &= (-1)^{s-1} 3\mu / \{(2s+1)(2s+3)\}.
 \end{aligned}$$

Hence

$$(P_{2s-1})_{r=a} = (-1)^{s-1} 3W a \cos \theta / \{(2s+1)(2s+3)\}, \quad (18)$$

and it follows from (8) that

$$\begin{aligned}
 c_{2s} &= -\frac{1}{\frac{4}{3}\pi a^3} \iint (-1)^{s-1} \frac{3W a}{(2s+1)(2s+3)} \cos^2 \theta dS, \quad s > 0 \\
 &= (-1)^s 3W / \{(2s+1)(2s+3)\}
 \end{aligned} \quad (19)$$

and c_{2s-1} is zero. Next

$$\begin{aligned}
 (\partial P_{2s-1} / \partial r)_{r=a} &= -W \{2Q_{2, 2s} + 4Q_{4, 2s} + \dots + 2m Q_{2m, 2s} + \dots + (2s+2) Q_{2s+2, 2s}\} \\
 &= (-1)^s \frac{3W}{4} \left[2 \frac{(1-\nu)^{s-1}}{s!} \left(\frac{d}{d\nu} \right)^{s-1} \left\{ \frac{2}{3} \nu^{(2s-1)/2} - \nu^{(2s-1)/2} (1-\nu) \right\} \right. \\
 &\quad - 4 \frac{(1-\nu)^{s-2}}{1! (s-1)!} \left(\frac{d}{d\nu} \right)^{s-1} \left\{ \frac{2}{5} \nu^{(2s-1)/2} (1-\nu) - \nu^{(2s-1)/2} (1-\nu)^2 \right\} \\
 &\quad \dots - (-1)^m 2m \frac{(1-\nu)^{s-m}}{(m-1)! (s-m+1)!} \times \\
 &\quad \left(\frac{d}{d\nu} \right)^{s-1} \left\{ \frac{2}{2m+1} \nu^{(2s-1)/2} (1-\nu)^{m-1} - \nu^{(2s-1)/2} (1-\nu)^m \right\} \\
 &\quad \dots + (-1)^s (2s+2) \frac{(1-\nu)^{-1}}{s!} \times \\
 &\quad \left. \left(\frac{d}{d\nu} \right)^{s-1} \left\{ \frac{2}{2s+3} \nu^{(2s-1)/2} (1-\nu)^s - \nu^{(2s-1)/2} (1-\nu)^{s+1} \right\} \right] \\
 &= (-1)^s \frac{3W}{4} \left[-\frac{4\nu^{1/2}}{(2s+1)(2s+3)} \times \right. \\
 &\quad \left\{ (1-\nu)^{s-1} + s^{-1} C_1 \nu (1-\nu)^{s-2} + s^{-1} C_2 \nu^2 (1-\nu)^{s-3} + \dots + \nu^{s-1} \right\} + 2\nu^{(2s-1)/2} (1-\nu) \Big] \\
 &= (-1)^s W \left[\frac{3}{2} \mu^{2s-1} (1-\mu^2) - 3\mu / \{(2s+1)(2s+3)\} \right], \quad s > 0.
 \end{aligned} \quad (20)$$

Finally

$$\begin{aligned}
 \left(\frac{\partial P_{2s-1}}{\partial z} \right)_{r=a} &= \mu \left(\frac{\partial P_{2s-1}}{\partial r} \right)_{r=a} + \frac{1-\mu^2}{a} \left(\frac{\partial P_{2s-1}}{\partial \mu} \right)_{r=a} \\
 &= (-1)^s W \left\{ \frac{3}{2} \mu^{2s} (1-\mu^2) - \frac{3\mu^2}{(2s+1)(2s+3)} - \frac{3(1-\mu^2)}{(2s+1)(2s+3)} \right\} \\
 &= (-1)^s W \left[\frac{3}{2} \mu^{2s} (1-\mu^2) - 3 / \{(2s+1)(2s+3)\} \right].
 \end{aligned}$$

Substituting now in equation (10), we have

$$\begin{aligned}
 & (-1)^s W \left[\frac{3}{2} \mu^{2s-1} (1 - \mu^2) - 3\mu / \{(2s+1)(2s+3)\} \right] \\
 & = \mu \left\{ (-1)^{s-1} \frac{3W}{(2s+1)(2s+3)} + (-1)^{s-2} \frac{3W}{(2s-1)(2s+1)} \right\} \\
 & \quad - \mu (-1)^{s-1} W \left\{ \frac{3}{2} \mu^{2s-2} (1 - \mu^2) - \frac{3}{(2s-1)(2s+1)} \right\}
 \end{aligned}$$

for $s = 2, 3, \dots$ and

$$\begin{aligned}
 (\partial P_1 / \partial r)_{r=a} &= -W \left\{ \frac{3}{2} \mu (1 - \mu^2) - \frac{3}{5} \mu \right\} = -W \left\{ \frac{1}{10} \mu - \frac{3}{2} \mu^3 \right\} \\
 &= \frac{\cos \theta}{\frac{4}{3} \pi a^3} \iint (P_1)_{r=a} \cos \theta dS - W \cos \theta - \cos \theta \left[\frac{\partial}{\partial z} \left\{ \frac{1}{2} W a \left(\frac{a}{r} \right)^2 \cos \theta \right\} \right]_{r=a}.
 \end{aligned}$$

It follows that the sequence P_{2s-1} , given in (13), satisfies the conditions of the problem, and therefore determines the motion.

Convergence.

§ 5. Before proceeding to the results obtained from this sequence, we shall consider the convergence of the series for u, v, w , and P , in terms of P_{2s-1} .

From the expression for $Q_{2m, 2s}(\mu)$, given in (14), it is seen that, for $m \leq s$

$$\begin{aligned}
 |Q_{2m, 2s}(\mu)| &< \frac{3}{2} \cdot \frac{1}{2^s (m-1)! (s-m+1)!} \times \\
 &\quad [2(2m+3)(2m+5) \dots (2s+2m-3) \{1 + {}^{m-1}C_1 + {}^{m-1}C_2 + \dots\} \\
 &\quad + (2m+3)(2m+5) \dots (2s+2m-1) \{1 + {}^mC_1 + {}^mC_2 + \dots\}] \\
 &< \frac{3}{2} \cdot \frac{(2m+3)(2m+5) \dots (2s+2m-3)}{2^{s-m} (m-1)! (s-m+1)!} (2s+2m) \\
 &< \frac{3}{2} 2^{m-1} \frac{(s+m)(s+m-1)(s+m-2) \dots (m+2)}{(m-1)! (s-m+1)!} \\
 &< \frac{3}{2} \frac{2^{m-1}}{(m-1)!} 2s(2s-1)(2s-2) \dots (s+2).
 \end{aligned}$$

Hence

$$\frac{|Q_{2m, 2s}(\mu)|}{(2s-1)!} < 3 \cdot \frac{2^{m-1}}{(m-1)!} \cdot \frac{s}{(s+1)!} \quad \text{for } m \leq s$$

and

$$|Q_{2s+2, 2s}(\mu)| \leq 3 / (4s+6).$$

Consequently

$$\frac{1}{Wa} \cdot \frac{|P_{2s-1}|}{(2s-1)!} < \frac{3}{s!} \sum_{m=1}^{s+1} \frac{2^{m-1}}{(m-1)!} < \frac{24}{s!}$$

and

$$\begin{aligned}
 |P| &\leq 2\omega \sum_{s=1}^{\infty} |P_{2s-1}| (2\omega t)^{2s-1} / (2s-1)! \\
 &< \sum_{s=1}^{\infty} 48Wa\omega (2\omega t)^{2s-1} / s!
 \end{aligned}$$

i.e. the series for P in terms of the sequence P_{2s-1} is absolutely convergent for finite values of t .

Similarly it may be shown that

$$|\text{Gradient of } P_{2s-1}| < (4s+3)|P_{2s-1}|,$$

and that the series for u, v, w , in terms of P_{2s-1} , are absolutely convergent for finite values of t .

The Motion of the Sphere.

§ 6. The velocity of the centre of the sphere is given by

$$V = \sum_{s=0}^{\infty} c_s (2\omega t)^s / s!$$

and from (19)

$$\left. \begin{aligned} c_{2s} &= (-1)^s 3W / \{(2s+1)(2s+3)\} \\ c_{2s+1} &= 0 \end{aligned} \right\}, \quad s \geq 0.$$

Hence

$$\begin{aligned} V &= \sum_{s=0}^{\infty} (-1)^s \frac{3W}{(2s+1)(2s+3)} \cdot \frac{(2\omega t)^{2s}}{(2s)!} \\ &= 3W \left\{ \frac{\sin 2\omega t}{(2\omega t)^3} - \frac{\cos 2\omega t}{(2\omega t)^2} \right\}. \end{aligned}$$

This may be expressed in a neater form by using the notation

$$S_n(x) = x^{n+1} \left(-\frac{1}{x} \cdot \frac{d}{dx} \right)^n \frac{\sin x}{x}$$

so that

$$V = 3W \cdot S_1(2\omega t) / (2\omega t)^2.$$

The values of V/W for $2\omega t = 0, 1, 2, \dots, 14$ have been found, and from them the Diagram I constructed, which shows the ratio V/W plotted against that

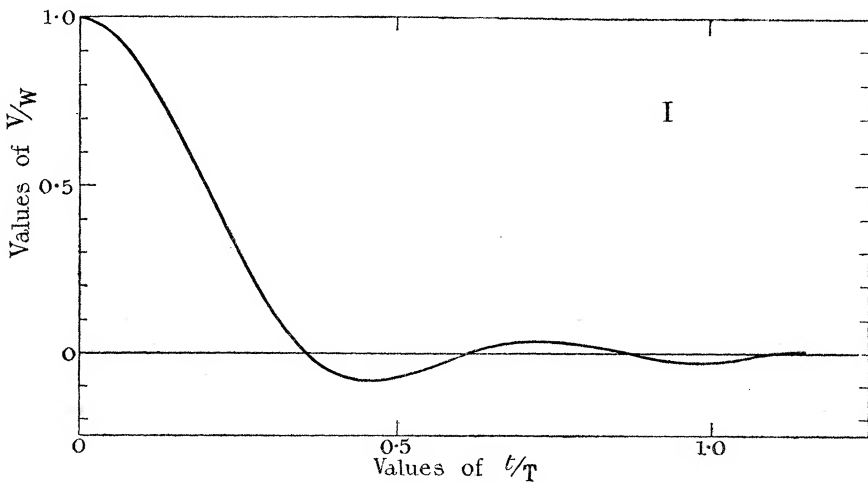


DIAGRAM I.

of t/T , where $T = 2\pi/\omega$ is the time of a revolution of the undisturbed liquid.

It is to be observed that the value of V is oscillatory, but tends rapidly to zero; the sphere oscillates about its final position with continually decreasing amplitude, the period of oscillation tending to a constant value equal to $\frac{1}{2}T$; after one revolution of the liquid the sphere is virtually at rest.

The distance between the initial and final positions of the sphere is given by

$$\int_0^\infty V dt = \int_0^\infty 3W \left\{ \frac{\sin 2\omega t}{(2\omega t)^3} - \frac{\cos 2\omega t}{(2\omega t)^2} \right\} dt = \frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{3W}{2\omega} = \frac{3}{16} WT.$$

The quantity WT is the distance the sphere would travel during the time T with its initial velocity of projection, consequently the sphere comes virtually to rest after a time T , at a point $3/16$ of this distance from the point of projection.

Velocity of the Liquid along the Axis of the Sphere.

§ 7. Along the axis of the sphere we have $\mu = \pm 1$, and taking first the case $\mu = 1$, we derive from (15)

$$Q_{2m, 2s} = 0, \quad m < s,$$

with

$$Q_{2s, 2s} = (-1)^{s-1} 3/(4s+2), \quad Q_{2s+2, 2s} = (-1)^s 3/(4s+6),$$

so that

$$P_{2s-1} = (-1)^s \frac{3Wa}{2} \left\{ \left(\frac{a}{r} \right)^{2s+2} \frac{1}{2s+3} - \left(\frac{a}{r} \right)^{2s} \frac{1}{2s+1} \right\}$$

and on the axis of the sphere

$$\frac{\partial P_{2s-1}}{\partial z} = \frac{\partial P_{2s-1}}{\partial r} = (-1)^{s+1} \frac{3W}{2} \left\{ \left(\frac{a}{r} \right)^{2s+3} \frac{2s+2}{2s+3} - \left(\frac{a}{r} \right)^{2s+1} \frac{2s}{2s+1} \right\}.$$

From equations (6) we have

$$w_s = -\partial P_{s-1} / \partial z, \quad s = 0, 1, 2, \dots,$$

hence it follows that

$$w_{2s+1} = 0$$

$$w_{2s} = -\frac{\partial P_{2s-1}}{\partial z} = (-1)^s \frac{3W}{2} \left\{ \left(\frac{a}{r} \right)^{2s+3} \frac{2s+2}{2s+3} - \left(\frac{a}{r} \right)^{2s+1} \frac{2s}{2s+1} \right\} \quad s \geq 0.$$

Since

$$w = \sum_{s=0}^{\infty} w_s (2\omega t)^s / s!$$

we have, on substituting the expressions obtained for the coefficients w_s ,

$$\begin{aligned}
 w &= \sum_{s=0}^{\infty} (-1)^s \frac{3W}{2} \left\{ \left(\frac{a}{r} \right)^{2s+3} \frac{2s+2}{2s+3} - \left(\frac{a}{r} \right)^{2s+1} \frac{2s}{2s+1} \right\} \frac{(2\omega t)^{2s}}{(2s)!} \\
 &= \frac{3W}{2} \left(\frac{a}{r} \right)^3 \sum_{s=0}^{\infty} (-1)^s \frac{2s+2}{2s+3} \frac{(2a\omega t/r)^{2s}}{(2s)!} \\
 &\quad + \frac{3W}{2} \frac{a}{r} \sum_{s=1}^{\infty} (-1)^{s-1} \frac{2s}{2s+1} \frac{(2a\omega t/r)^{2s}}{(2s)!} \\
 &= \frac{3W}{2} \left(\frac{a}{r} \right)^3 \left\{ \cos(2a\omega t/r) - \frac{\sin(2a\omega t/r)}{2a\omega t/r} - 2 \frac{\cos(2a\omega t/r)}{(2a\omega t/r)^2} \right. \\
 &\quad \left. + 2 \frac{\sin(2a\omega t/r)}{(2a\omega t/r)^3} \right\} - \frac{3W}{2} \frac{a}{r} \left\{ \cos(2a\omega t/r) - \frac{\sin(2a\omega t/r)}{(2a\omega t/r)} \right\} \\
 &= \frac{3W}{2} \left\{ \frac{a}{r} - \left(\frac{a}{r} \right)^3 + 2 \frac{a}{r} \frac{1}{(2\omega t)^2} \right\} S_1 \left(\frac{2a\omega t}{r} \right), \tag{21}
 \end{aligned}$$

which, on $r = a$, agrees with the expression obtained for the velocity of the sphere.

The case $\mu = -1$ gives the same expression for w , and, on the axis of the sphere,

$$u = v = 0.$$

Perhaps the clearest way to discuss the function w is to examine the variation with r for constant values of t . A table has been constructed and from it the Diagram II formed, in which the ratio w/W is plotted against that of r/a ; the curves are numbered 0, 1, 2, ... 10, and represent w/W at

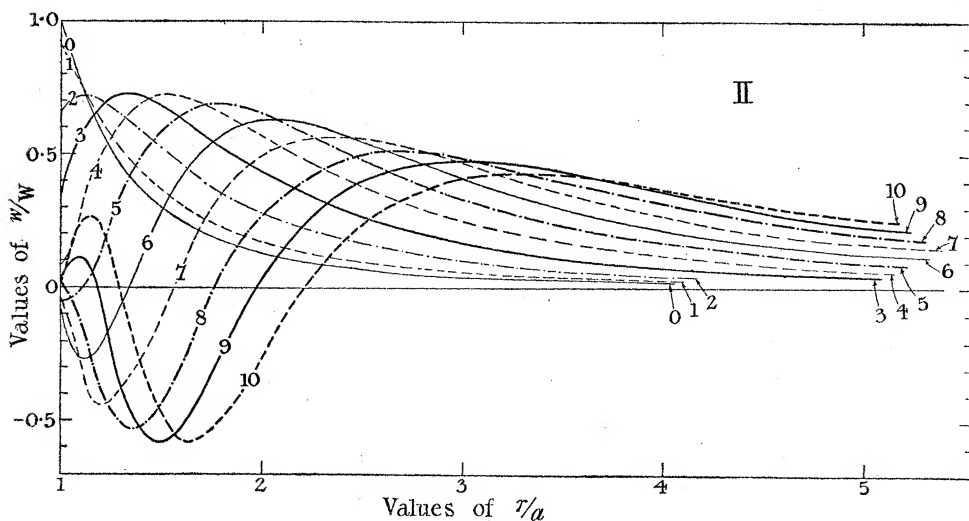


DIAGRAM II.

points on the axis of the sphere for $2\omega t = 0, 1, 2, \dots, 10$ respectively. It may be noted that $2\omega t = 10$ corresponds roughly to $t = 0.8T$.

A further Diagram III has been drawn showing the velocities at times corresponding to $T, 2T, 3T$, respectively.

It may be noted that the function $S_1(2a\omega t/r)$ oscillates between bounds in

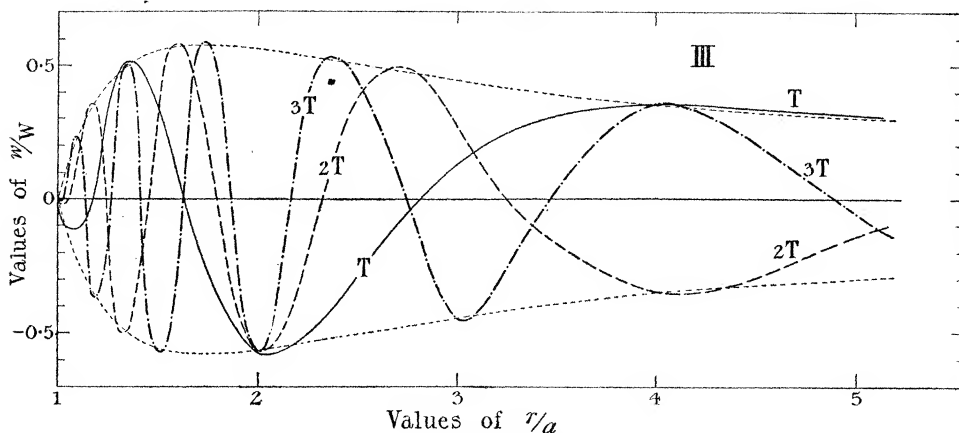


DIAGRAM III.

the neighbourhood of ± 1 . Referring now to the Diagrams II and III and to the expression for w given in (21), it is seen that w is always less than W . As r increases, w is ultimately positive and tends to zero. At intermediate points the velocity may be positive or negative, the disturbance being a maximum in the neighbourhood of $r = 1.7a$. Here it is to be noted, however, that as t increases the velocity gradient along the axis becomes large, due to the fact that in $\partial w / \partial r$ there is a factor $2\omega t$. This point is clearly illustrated in Diagram III, which indicates the state of the liquid as t becomes large; the changes of velocity are observed to be contained within upper and lower bounds, which are $\pm \frac{3}{2} W \{a/r - (a/r)^3\}$, since for t large we have approximately

$$w = \frac{3}{2} W \left(\frac{a}{r} - \frac{a^3}{r^3} \right) S_1 \left(\frac{2a\omega t}{r} \right).$$

Velocity of the Liquid on the Equatorial Plane of the Sphere.

§ 8. The equatorial plane of the sphere is the plane defined by $\mu = 0$. Here we have

$$u = v = 0,$$

so that the velocity is wholly parallel to the axis of rotation. We have

$$\left(\frac{\partial P_{2s-1}}{\partial z} \right)_{\mu=0} = \frac{1}{r} \left(\frac{\partial P_{2s-1}}{\partial \mu} \right)_{\mu=0}$$

and from (14)

$$\begin{aligned} \left(\frac{\partial Q_{2m, 2s}}{\partial \mu} \right)_{\mu=0} &= (-1)^{s-m} \frac{3}{2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2s-1)}{2^s (m-1)! (s-m+1)!} \left\{ \frac{2}{2m+1} - 1 \right\} \quad s, m > 0 \\ &= (-1)^{s-m+1} \frac{3}{2} \cdot \frac{2m-1}{2m+1} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2s-1)}{2^s (m-1)! (s-m+1)!} \end{aligned}$$

so that

$$\begin{aligned} \left(\frac{\partial P_{2s-1}}{\partial z} \right)_{\mu=0} &= (-1)^s \frac{3W}{2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2s-1)}{2^s} \times \\ &\quad \left\{ \frac{1}{3} \cdot \frac{1}{s!} \left(\frac{a}{r} \right)^3 - \frac{3}{5} \frac{1}{(s-1)! 1!} \left(\frac{a}{r} \right)^5 + \frac{5}{7} \frac{1}{(s-2)! 2!} \left(\frac{a}{r} \right)^7 - \dots \right. \\ &\quad \dots - (-1)^m \frac{2m-1}{2m+1} \frac{1}{(s-m+1)! (m-1)!} \left(\frac{a}{r} \right)^{2m+1} + \dots \\ &\quad \left. \dots + (-1)^s \frac{2s+1}{2s+3} \cdot \frac{1}{s!} \left(\frac{a}{r} \right)^{2s+3} \right\} \\ &= (-1)^s \frac{3W}{2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2s-1)}{2^s \cdot s!} \times \\ &\quad \left\{ \frac{1}{3} \left(\frac{a}{r} \right)^3 - {}^s C_1 \frac{3}{5} \left(\frac{a}{r} \right)^5 + {}^s C_2 \frac{5}{7} \left(\frac{a}{r} \right)^7 - \dots \right. \\ &\quad \left. - (-1)^m {}^s C_{m-1} \frac{2m-1}{2m+1} \left(\frac{a}{r} \right)^{2m+1} + \dots + (-1)^s \frac{2s+1}{2s+3} \left(\frac{a}{r} \right)^{2s+3} \right\}. \end{aligned} \quad (22)$$

The quantity within the brackets, on putting $a/r = \lambda$, becomes

$$\begin{aligned} &\frac{1}{3} \lambda^3 - {}^s C_1 \frac{3}{5} \lambda^5 + {}^s C_2 \frac{5}{7} \lambda^7 - \dots + (-1)^s \frac{2s+1}{2s+3} \lambda^{2s+3} \\ &= \int_0^\lambda \lambda^2 (1-\lambda^2)^s d\lambda - 2s \int_0^\lambda \lambda^4 (1-\lambda^2)^{s-1} d\lambda \\ &= \lambda^3 (1-\lambda^2)^s - 2 \int_0^\lambda \lambda^2 (1-\lambda^2)^s d\lambda. \end{aligned} \quad (23)$$

Hence

$$\begin{aligned} \left(\frac{\partial P_{2s-1}}{\partial z} \right)_{\mu=0} &= (-1)^s \frac{3W}{2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2s-1)}{2^s \cdot s!} \times \\ &\quad \left\{ \lambda^3 (1-\lambda^2)^s - 2 \int_0^\lambda \lambda^2 (1-\lambda^2)^s d\lambda \right\}, \end{aligned}$$

so that, on the equatorial plane,

$$\begin{aligned} w &= \sum_{s=0}^{\infty} w_s (2\omega t)^s / s! \\ &= -\frac{1}{2} W \lambda^3 - \sum_{s=1}^{\infty} (-1)^s \frac{3W}{2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2s-1)}{2^s \cdot s!} \times \\ &\quad \left\{ \lambda^3 (1-\lambda^2)^s - 2 \int_0^\lambda \lambda^2 (1-\lambda^2)^s d\lambda \right\} \frac{(2\omega t)^{2s}}{(2s)!}. \end{aligned}$$

Further,

$$\sum_{s=1}^{\infty} (-1)^s \frac{1 \cdot 3 \cdot 5 \dots (2s-1)}{2^s \cdot s!} (1-\lambda^2)^s \frac{(2\omega t)^{2s}}{(2s)!}$$

$$= \sum_{s=1}^{\infty} (-1)^s \frac{\{2\omega t \sqrt{(1-\lambda^2)}\}^{2s}}{2^{2s} (s!)^2} = J_0 \{2\omega t \sqrt{(1-\lambda^2)}\} - 1,$$

$J_0 \{2\omega t \sqrt{(1-\lambda^2)}\}$ being the Bessel function of zero order. Hence we have

$$w = -\frac{1}{2} W \lambda^3 - \frac{3}{2} W \left[\lambda^3 J_0 \{2\omega t \sqrt{(1-\lambda^2)}\} \right.$$

$$\left. - \lambda^3 - 2 \int_0^\lambda [\lambda^2 J_0 \{2\omega t \sqrt{(1-\lambda^2)}\} - \lambda^2] d\lambda \right]$$

$$= -\frac{3}{2} W \left[\lambda^3 J_0 \{2\omega t \sqrt{(1-\lambda^2)}\} - 2 \int_0^\lambda \lambda^2 J_0 \{2\omega t \sqrt{(1-\lambda^2)}\} d\lambda \right]. \quad (24)$$

This is the most compact expression which has been obtained for the velocity at points on the equatorial plane; it is convenient in that it enables us to obtain an approximation to the value of w .

When λ is small, the value of the integral is approximately $\frac{1}{3} \lambda^3 J_0(2\omega t)$, provided $2\omega t$ is not large, which gives w in the form

$$-\frac{1}{2} W \lambda^3 J_0(2\omega t).$$

To approximate to the value of w for t large, we use the well-known fact that, for x large, $J_0(x)$ is approximately

$$\sqrt{\left(\frac{2}{\pi x}\right)} \cos\left(x - \frac{\pi}{4}\right)$$

so that, provided $\sqrt{(1-\lambda^2)}$ is finite and $2\omega t$ sufficiently large, w becomes

$$-\frac{3W\lambda^3}{2} \sqrt{\left\{\frac{2}{\pi 2\omega t \sqrt{(1-\lambda^2)}}\right\}} \cos\left\{2\omega t \sqrt{(1-\lambda^2)} - \frac{\pi}{4}\right\}$$

$$+ 3W \int_0^\lambda \lambda^2 \sqrt{\left\{\frac{2}{\pi 2\omega t \sqrt{(1-\lambda^2)}}\right\}} \cos\left\{2\omega t \sqrt{(1-\lambda^2)} - \frac{\pi}{4}\right\} d\lambda,$$

and, on examination, we see that ultimately the integral is of a higher order of small quantities, so that w is approximately

$$-\frac{3W\lambda^3}{2} \sqrt{\left\{\frac{2}{\pi 2\omega t \sqrt{(1-\lambda^2)}}\right\}} \cos\left\{2\omega t \sqrt{(1-\lambda^2)} - \frac{\pi}{4}\right\}.$$

It follows, therefore, that there is ultimately an oscillation about the equatorial plane, the amplitude at any point continually decreasing and tending to zero, the period, however, tending to the constant value

$$\frac{2\pi}{2\omega \sqrt{(1-\lambda^2)}} = \frac{1}{2} \cdot \frac{T_1}{\sqrt{\{1-(a/r)^2\}}}.$$

Again, since $\partial w / \partial r$ contains a factor $(2\omega t)^{\frac{1}{2}}$, the velocity gradient ultimately increases without limit.

To examine further the equatorial velocity, we put the expression (24) into a power series in λ . To obtain this form we return to the expression for $(\partial P_{2s-1}/\partial z)_{\mu=0}$ given by (22) and consider the coefficient of λ^{2m+3} for $m > 0$ in

$$\sum_{s=1}^{\infty} \left(\frac{\partial P_{2s-1}}{\partial z} \right)_{\mu=0} \frac{(2\omega t)^{2s}}{2s!}.$$

It is

$$\begin{aligned} & (-1)^m \frac{2m+1}{2m+3} \cdot \frac{3W}{2} \sum_{s=m}^{\infty} (-1)^s C_m \frac{(2\omega t)^{2s}}{2^{2s} (s!)^2} \\ &= \frac{2m+1}{2m+3} \cdot \frac{1}{2^m m!} \cdot \frac{3W}{2} (2\omega t)^m \sum_{p=0}^{\infty} (-1)^p \frac{(2\omega t)^{2p+m}}{2^{2p+m} p! (p+m)!} \\ &= \frac{3W}{2} \cdot \frac{2m+1}{2m+3} \cdot \frac{1}{2^m m!} (2\omega t)^m J_m(2\omega t), \end{aligned}$$

and the coefficient of λ^3 is

$$-\frac{1}{2} W \{1 - J_0(2\omega t)\}.$$

Hence w is given by

$$\begin{aligned} & -\frac{3}{2} W \left\{ \frac{1}{3} \lambda^3 J_0(2\omega t) + \frac{3}{5} \cdot \frac{1}{2} \lambda^5 2\omega t J_1(2\omega t) + \frac{5}{7} \cdot \frac{1}{2 \cdot 4} \lambda^7 (2\omega t)^2 J_2(2\omega t) + \dots \right. \\ & \quad \left. \dots + \frac{2m+1}{2m+3} \frac{1}{2 \cdot 4 \cdot 6 \dots 2m} \lambda^{2m+3} (2\omega t)^m J_m(2\omega t) + \dots \right\}. \end{aligned}$$

From this expression we are able to construct a Table giving the values of w for $2\omega t = 0, 1, 2, \dots, 10$, and to graph the function w for constant values of t (Diagram IV). The Diagrams II, III, IV are all drawn to the

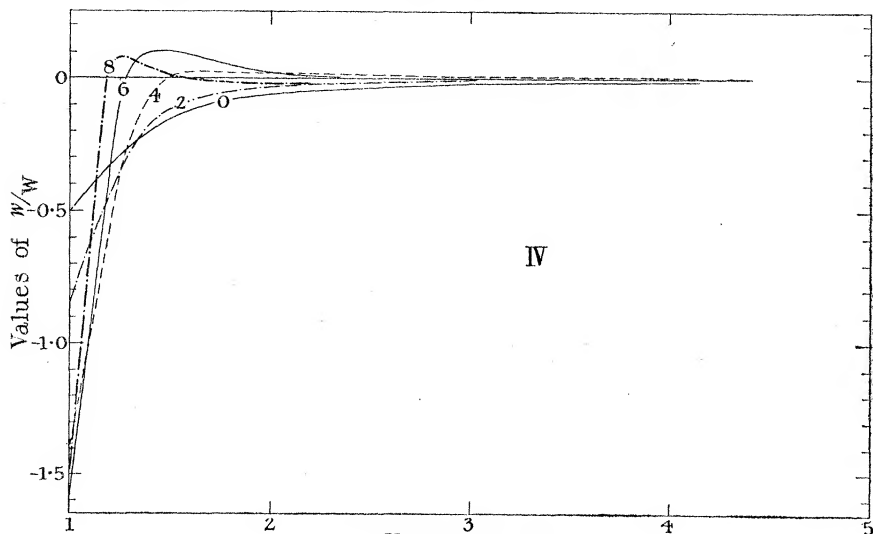


DIAGRAM IV.

same scale for purposes of comparison, but, in the last case, only the curves corresponding to $2\omega t = 0, 2, 4, 6, 8$ are shown, since the variation is so small.

An examination of Diagram IV confirms the results already given; it also indicates that the disturbance on the equatorial plane is practically confined to the liquid in the immediate neighbourhood of the sphere.

On the sphere itself, putting $\lambda = 1$ in (23),

$$\begin{aligned} \left(\frac{\partial P_{2s-1}}{\partial z}\right)_{\mu=0} &= (-1)^{s+1} \frac{3W}{2} \frac{1.3.5 \dots (2s-1)}{2^s \cdot s!} 2 \cdot \frac{1}{2s+3} \cdot \frac{2s}{2s+1} \cdot \frac{2s-2}{2s-1} \dots \frac{2}{3} \\ &= (-1)^{s+1} 3W / \{(2s+1)(2s+3)\}, \end{aligned}$$

so that, on the equator of the sphere,

$$\begin{aligned} w &= -\frac{W}{2} - 3W \sum_{s=1}^{\infty} (-1)^{s+1} \frac{1}{(2s+1)(2s+3)} \cdot \frac{(2\omega t)^{2s}}{(2s)!} \\ &= -\frac{3}{2} W + 3W \cdot S_1(2\omega t)/(2\omega t)^2, \end{aligned}$$

which rapidly tends to the limiting value $-\frac{3}{2} W$.

Velocity of the Liquid at Points on the Sphere.

§ 9. To discuss the velocity of the liquid at points on the surface of the sphere, we transform the dynamical equations (3) into spherical polar co-ordinates, and obtain

$$\frac{\partial u'}{\partial t} - 2\omega v' \cos \theta = -\frac{\partial P}{r \partial \theta},$$

$$\frac{\partial v'}{\partial t} + 2\omega w' \sin \theta + 2\omega u' \cos \theta = 0, \quad (25)$$

$$\frac{\partial w'}{\partial t} - 2\omega v' \sin \theta = -\frac{\partial P}{\partial r}, \quad (26)$$

where w' coincides with w for $\theta = 0$ and $-u'$ coincides with w on the equator.

On the sphere, we have

$$(w')_{r=a} = V \cos \theta = 3W \cos \theta \left\{ \frac{\sin 2\omega t}{(2\omega t)^3} - \frac{\cos 2\omega t}{(2\omega t)^2} \right\}$$

and from (20)

$$\left(\frac{\partial P_{2s-1}}{\partial r}\right)_{r=a} = (-1)^s W \left\{ \frac{3}{2} \cos^{2s-1} \theta \sin^2 \theta - \frac{3 \cos \theta}{(2s+1)(2s+3)} \right\} \quad s > 0,$$

so that

$$\begin{aligned}
 \left(\frac{\partial P}{\partial r} \right)_{r=a} &= 2\omega \sum_{s=0}^{\infty} \left(\frac{\partial P_s}{\partial r} \right)_{r=a} \frac{(2\omega t)^s}{s!} \\
 &= 2\omega \sum_{s=1}^{\infty} (-1)^s W \left\{ \frac{3}{2} \cos^{2s-1} \theta \sin^2 \theta - \frac{3 \cos \theta}{(2s+1)(2s+3)} \right\} \frac{(2\omega t)^{2s-1}}{(2s-1)!} \\
 &= -2\omega W \left[\frac{3}{2} \sin^2 \theta \sin(2\omega t \cos \theta) \right. \\
 &\quad \left. + 3 \cos \theta \left\{ \frac{\sin 2\omega t}{(2\omega t)^2} + \frac{3 \cos 2\omega t}{(2\omega t)^3} - \frac{3 \sin 2\omega t}{(2\omega t)^4} \right\} \right].
 \end{aligned}$$

Hence from (26)

$$(v')_{r=a} = -\frac{3}{2} W \sin \theta \sin(2\omega t \cos \theta),$$

and from (25)

$$(-u')_{r=a} = -\frac{3}{2} W \sin \theta \cos(2\omega t \cos \theta) + 3 W \sin \theta \cdot S_1(2\omega t)/(2\omega t)^2. \quad (27)$$

The expression for $(w')_{r=a}$ has already been treated; it is simply the radial component of the velocity of the sphere.

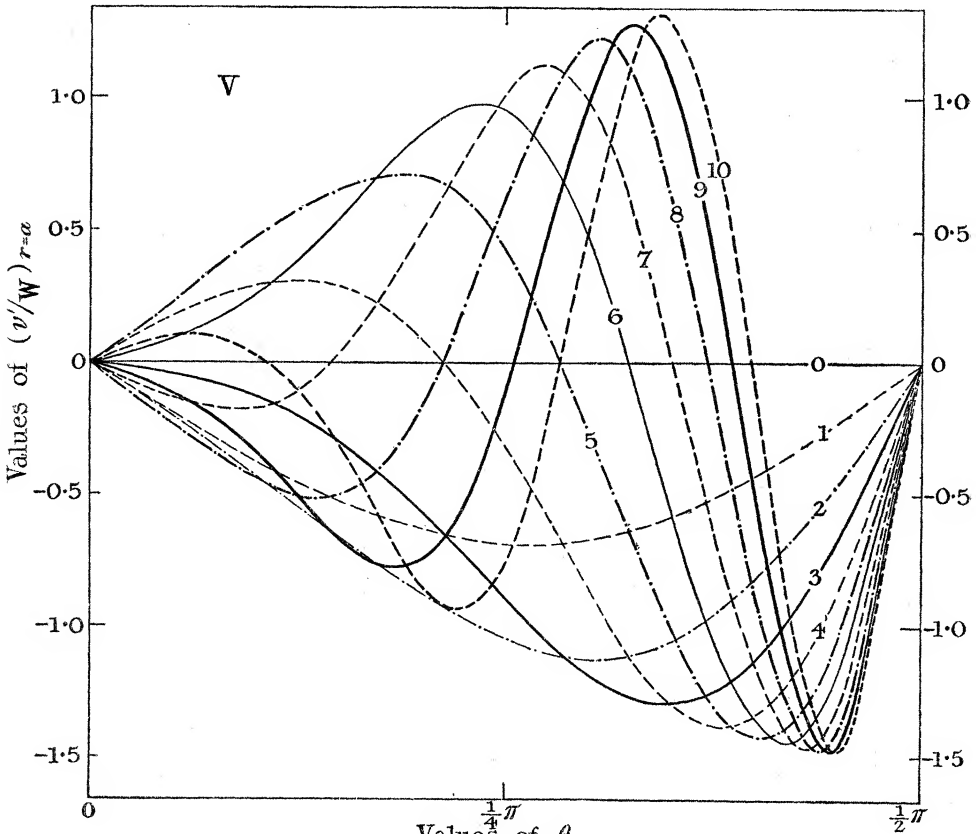


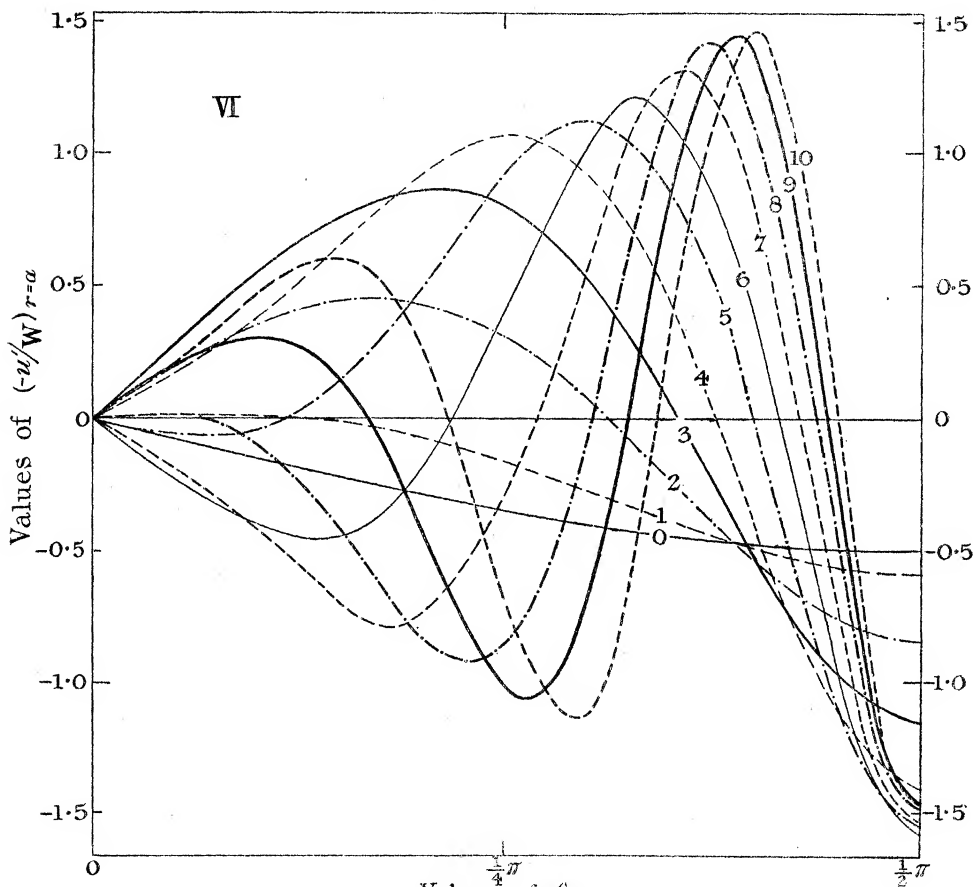
DIAGRAM V.

Considering the expression for $(v')_{r=a}$, a Table may be formed giving its value for θ between 0 and $\frac{1}{2}\pi$ at times corresponding to $2\omega t = 0, 1, 2, \dots 10$ respectively, and from it the Diagram V constructed. The values for θ from $\frac{1}{2}\pi$ to π are those from 0 to $\frac{1}{2}\pi$, repeated in the reverse order with change of sign.

It is observed that the transverse velocity, $(v')_{r=a}$, oscillates between bounds $\pm \frac{3}{2}W \sin \theta$, which are independent of the time.

The velocity along a meridian, coinciding with w on the equator, is given by (27). The variation of this quantity is indicated by the Diagram VI, which is constructed similarly to V. The values for θ from $\frac{1}{2}\pi$ to π are those from 0 to $\frac{1}{2}\pi$ in the reverse order.

The value of $(-u')_{r=a}$ oscillates between bounds in the neighbourhood of $\pm \frac{3}{2}W \sin \theta$, approaching these bounds more nearly as t increases; the



Values of θ
DIAGRAM VI.

irregularity is due to the term $3W \sin \theta \cdot S_1 (2\omega t)/(2\omega t)^2$, which rapidly tends to zero.

Hence, for t large, the velocity of the liquid tangential to the sphere at a point of co-latitude θ , has components

$$-\frac{3}{2} W \sin \theta \cos (2\omega t \cos \theta), \quad -\frac{3}{2} W \sin \theta \sin (2\omega t \cos \theta),$$

which combine into $\frac{3}{2} W \sin \theta$ in a direction making an angle α , measured in the negative direction from the meridian (θ increasing) through the point where $\alpha = 2\omega t \cos \theta$.

Hence, ultimately, the velocity of the liquid tangential to the sphere at any point is constant in magnitude and equal to $\frac{3}{2} W \sin \theta$, but changes its direction with constant angular speed $2\omega \cos \theta$, which is a maximum at the poles and zero at the equator.

A point to notice, however, is the fact that $(\partial v'/\partial \theta)_{r=a}$ contains the factor $2\omega t$, and therefore ultimately increases without limit. This means that we reach a stage beyond which the components of vorticity can no longer be regarded as small. A similar deduction is obtained from $\partial w/\partial r$ over the equatorial plane, which contains the factor $(2\omega t)^{\frac{1}{2}}$. A physical restriction to the solution of the problem is thus introduced.
